$y = e^{-2x}$ is a solution of the homogeneous differential equation $(2x+1)y'' + 4xy' - 4y = 0$. **SCORE: _____ / 40 PTS**

Find the general solution of the non-homogeneous differential equation $(2x+1)y'' + 4xy' - 4y = 2e^{-2x}(2x+1)^2$.

$$y_2 = ve^{-2x}$$

$$y_2' = v'e^{-2x} - 2ve^{-2x}$$

$$y_2'' = v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}$$

$$e^{-2x}\,[\,v''(2x+1) + v'(-4(2x+1)) + v(4(2x+1)) \quad 3$$

$$+v'(4x) \quad + v(-2(4x)) \quad 2$$

$$+v(-4)\,] = 0 \quad 1$$

$$\cancel{e^{-2x}}\,[\,v''(2x+1) + v'(-4x-4)\,] = 0 \quad 2$$

LET $u = v'$

$$u'(2x+1) + u(-4x-4) = 0$$

$$\int \frac{1}{u}\,du = \int \frac{4x+4}{2x+1}\,dx \quad 2$$

$$= \int \left(2 + \frac{2}{2x+1}\right) dx \quad 2$$

$$\ln|u| = 2x + \ln|2x+1| \quad 2$$

$$v' = u = (2x+1)e^{2x} \quad 2$$

$$v = xe^{2x} \quad 2$$

$$y_2 = x \quad 2$$

$2x+1$	$+\; e^{2x}$
2	$-\; \frac{1}{2}e^{2x}$
0	$\frac{1}{4}e^{2x}$

$$W = \begin{vmatrix} e^{-2x} & x \\ -2e^{-2x} & 1 \end{vmatrix} = (1+2x)e^{-2x} \quad 2 \quad 3$$

$$g = \frac{2e^{-2x}(2x+1)^2}{2x+1} = 2e^{-2x}(2x+1) \quad 3$$

$$y_p = -e^{-2x}\int \frac{2\cancel{e^{-2x}}\cancel{(2x+1)}x}{\cancel{(1+2x)}\cancel{e^{-2x}}}\,dx + x\int \frac{2e^{-2x}\cancel{(2x+1)}\cancel{e^{-2x}}}{\cancel{(1+2x)}\cancel{e^{-2x}}}\,dx \quad 2 \quad 2$$

$$= -e^{-2x}\int 2x\,dx + x\int 2e^{-2x}\,dx$$

$$= -x^2e^{-2x} - xe^{-2x} \quad 3 \quad 3$$

$$y = -(x^2+x)e^{-2x} + Ae^{-2x} + Bx \quad 2$$

$y = 4\sqrt{x}$ is a particular solution of the non-homogeneous differential equation $9x^2y'' - 3xy' + 4y = \sqrt{x}$. **SCORE: _____ / 15 PTS**

Solve the initial value problem $9x^2y'' - 3xy' + 4y = 2\sqrt{x}, \quad y(1) = -7, \quad y'(1) = 5$.

(2) $9r^2 - 12r + 4 = 0$

$(3r-2)^2 = 0$

(1) $r = \frac{2}{3}, \frac{2}{3}$

$y = 2(4\sqrt{x}) + Ax^{\frac{2}{3}} + Bx^{\frac{2}{3}}\ln x$

$= 8\sqrt{x} + Ax^{\frac{2}{3}} + Bx^{\frac{2}{3}}\ln x$ (2) (2) (2)

$y(1) = 8 + A = -7 \longrightarrow A = -15$ (1)

$y' = 4x^{-\frac{1}{2}} + \frac{2}{3}Ax^{-\frac{1}{3}} + \frac{2}{3}Bx^{-\frac{1}{3}}\ln x + Bx^{-\frac{1}{3}}$ (2)

$y'(1) = 4 + \frac{2}{3}(-15) + B = 5$

$B = 11$ (2)

$y = 8\sqrt{x} - 15x^{\frac{2}{3}} + 11x^{\frac{2}{3}}\ln x$ (1)

Use elimination (as shown in lecture) to solve the system

SCORE: _____ / 55 PTS

$$(2D-4)(x)-(D-2)(y)=4e^{-2t}$$
$$(D+7)(x)+(2D-1)(y)=-5e^{-2t}$$

④

$\ominus$ $(D+7)(2D-4)(x)-(D+7)(D-2)(y) = -8e^{-2t}+28e^{-2t}=20e^{-2t}$ ②

$(2D-4)(D+7)(x)+(2D-4)(2D-1)(y) = 20e^{-2t}+20e^{-2t}=40e^{-2t}$ ②

④ $(5D^2-5D-10)(y)= 20e^{-2t}$ ②

$(D^2-D-2)(y)= 4e^{-2t}$

$r^2-r-2=0 \rightarrow r=2,-1 \rightarrow y_h = Ae^{2t}+Be^{-t}$

$y_p = Ke^{-2t}$ ②

$y_p' = -2Ke^{-2t}$

$y_p'' = 4Ke^{-2t}$

④

$y_p''-y_p'-2y_p = e^{-2t}(4K+2K-2K) = 4Ke^{-2t} = 4e^{-2t}$

$K=1$

③ ②

$y = Ae^{2t}+Be^{-t}+e^{-2t}$

④

$\oplus$ $(2D-1)(2D-4)(x)-(2D-1)(D-2)(y) = -16e^{-2t}-4e^{-2t}=-20e^{-2t}$ ②

$(D-2)(D+7)(x)+(D-2)(2D-1)(y) = 10e^{-2t}+10e^{-2t}=20e^{-2t}$ ②

④ $(5D^2-5D-10)(x)=0$

$x = Ce^{2t}+De^{-t}$ ③

$(D+7)(x)+(2D-1)(y)$

$= 2Ce^{2t}-De^{-t}$

$+7Ce^{2t}+7De^{-t}$

$+4Ae^{2t}-2Be^{-t}-4e^{-2t}$

$-Ae^{2t}-Be^{-t}-e^{-2t}$

⑥

$= (9C+3A)e^{2t}+(6D-3B)e^{-t}-\cancel{5e^{-2t}} = \cancel{-5e^{-2t}}$

$9C+3A=0 \qquad 6D-3B=0$

$A=-3C$ ③ $\qquad B=2D$ ③

③
$$\begin{aligned} x &= Ce^{2t}+De^{-t} \\ y &= -3Ce^{2t}+2De^{-t}+e^{-2t} \end{aligned}$$

Find the general solution of $y''' - y'' + y' - y = x + 4\cos x$. **SCORE: _____ / 40 PTS**

$$r^3 - r^2 + r - 1 = 0$$
$$r^2(r-1) + (r-1) = 0$$
$$(r^2+1)(r-1) = 0 \quad 2$$
$$r = \pm i, 1 \quad 1$$
$$y_h = A\cos x + B\sin x + Ce^x \quad 2$$
$$y_p = Dx\cos x + Fx\sin x + Gx + H \quad 2$$
$$y_p' = D\cos x - Dx\sin x + G \quad 4$$
$$+ Fx\cos x + F\sin x$$
$$= (Fx+D)\cos x + (-Dx+F)\sin x + G \quad 1$$
$$y_p'' = F\cos x + (-Fx-D)\sin x$$
$$+ (-Dx+F)\cos x - D\sin x$$
$$= (-Dx+2F)\cos x + (-Fx-2D)\sin x \quad 4$$
$$y_p''' = -D\cos x + (Dx-2F)\sin x$$
$$+ (-Fx-2D)\cos x - F\sin x$$
$$= (-Fx-3D)\cos x + (Dx-3F)\sin x \quad 4$$

$$y_p''' - y_p'' + y_p' - y_p = (-Fx-3D+Dx-2F+Fx+D-Dx)\cos x$$
$$+ (Dx-3F+Fx+2D-Dx+F-Fx)\sin x$$
$$+ G - Gx - H$$
$$= (-2D-2F)\cos x + (-2F+2D)\sin x \quad 4$$
$$-Gx + G - H \quad 2$$

$$-2D-2F = 4 \qquad -G = 1 \rightarrow G = -1 \quad 1\tfrac{1}{2}$$
$$2D-2F = 0 \qquad G - H = 0 \quad H = -1 \quad 1\tfrac{1}{2}$$
$$-4F = 4 \rightarrow F = -1 \quad 1\tfrac{1}{2}$$
$$D = -1 \quad 1\tfrac{1}{2}$$

$$y = -x\cos x - x\sin x - x - 1 + A\cos x + B\sin x + Ce^x$$
$2 \qquad 2$